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On the Vector Bundles Whose Endomorphisms Yield Azumaya Algebras of Cyclic Type

HAJIME KAJI

Department of Mathematics, School of Science and Engineering, Waseda University, Ohkubo, Shinjuku, Tokyo 160, Japan

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INTRODUCTION

Let X be a non-singular projective variety over an algebraically closed field k with arbitrary characteristic p, let n be a positive integer prime to p, and let us consider the following diagram of étale cohomology sets:

$$\begin{tikzcd} & & H^1(X, \mathbb{G}_m) \arrow[d, "c_X"] \\ & H^1(X, \mu_n) \times H^1(X, \mu_n) \arrow[r, "\cup"] \arrow[d, "A"] & H^2(X, \mu_n) \arrow[d] \\ H^1(X, GL_n) \arrow[r, "\mathscr{E}nd"] & H^1(X, PGL_n) \arrow[r, "d_n"] & H^2(X, \mathbb{G}_m) \end{tikzcd} \tag{FD}$$

The definition of each map above is this: The lower horizontal sequence is induced from a well-known, fundamental sequence of étale sheaves of group schemes over X

$$1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1, \tag{FS}$$

so this sequence is exact. The right vertical sequence is induced from the Kummer sequence for the étale topology over X

$$1 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 1, \tag{KS}$$

so this sequence is also exact. The upper horizontal map $\cup$ is (non-canonically) defined by the cup-product on X with a fixed primitive nth root ζ of unity in k. The left vertical map A is defined by a generalization of the construction of a cyclic algebra over a field to the case over a scheme, which makes the diagram above commutative. We shall give a construction

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of an Azumaya algebra, denoted by $A(L, M)$, of rank n^2 over X from a pair of n-torsion line bundles L and M over X such that the diagram above commutes (see Sections 1 and 2).

We here note that (see, e.g., [1, Sect. 4; 7, III, IV]): $H^1(X, GL_n)$ is equal to the set of isomorphism classes of vector bundles of rank n over X (for the Zariski topology), in particular, $H^1(X, \mathbb{G}_m)$ coincides with the Picard group $\mathrm{Pic}(X)$ of X; $H^1(X, \mu_n)$ coincides with its n-torsion part $\mathrm{Pic}(X)_n$; $H^1(X, PGL_n)$ is equal to the set of isomorphism classes of Azumaya algebras of rank n^2 over X, whose elements correspond bijectively with the isomorphism classes of fibre bundles over X for the étale topology with a geometric fibre $\mathbb{P}^{n-1}$, namely, *projective space bundles of rank n* over X, via the functor of certain left ideals of the algebra.

Now, for a pair of n-torsion line bundles L and M over X, the commutativity and the exactness of the diagram above imply the equivalence of the following conditions:

(1) The Azumaya algebra $A(L, M)$ is isomorphic to $\mathcal{E}nd(V)$ for some vector bundle V over X.

(2) The cup-product $L \cup M$ is equal to $c_X(Z)$ for some line bundle Z over X.

So, one may expect that there would exist some relation between V and Z above. We here propose to discuss the following problem:

How can one construct the vector bundle V from the line bundle Z?

where one should note that V is uniquely determined by $A(L, M)$ up to tensoring line bundles over X.

The purpose of this article is to give an answer to this problem in case X is a product of two elliptic curves and $n = 2$. Namely, in this case, we shall construct *all* such vector bundles V from the line bundles Z.

In order to describe all the vector bundles V and to state our results, we need a "composition of vector bundles", which is the one corresponding to the distributive law of the cup-product on X. Precisely speaking, for a general integer n, this is defined modulo $\mathrm{Pic}(X)$ as follows: If, for n-torsion line bundles L_1, L_2 and M over X, the Azumaya algebras $A(L_1, M)$, $A(L_2, M)$ are isomorphic to $\mathcal{E}nd(V_1)$, $\mathcal{E}nd(V_2)$ for some vector bundles V_1, V_2 over X, respectively, then we define the *composition* of V_1 and V_2 to be a vector bundle V_{12} such that $A(L_1 \otimes L_2, M)$ is isomorphic to $\mathcal{E}nd(V_{12})$ (see Definition 2.8). We show the existence of the composition. Moreover, in case $n = 2$, we shall give its construction (see Section 4).

Using this terminology, one of the results is roughly stated as follows (for details, see Section 7): Our vector bundles V over a product X of two elliptic curves can be classsified into three types modulo $\mathrm{Pic}(X)$ as follows:

(1) A direct sum $\mathcal{O}_X \oplus L$, or $\mathcal{O}_X \oplus M$.

(2) A pullback of an indecomposable vector bundle over an elliptic curve by a morphism defined by the line bundle Z.

(3) A composition of two vector bundles of type (2) above.

As a consequence of our results, we obtain a criterion for a quadratic form of a certain type over the function field of X to have a rational solution. Precisely speaking, the quadratic form is the one induced from the reduced norm of our Azumaya algebra of rank 2^2, namely, *quaternion algebra*, over X. Moreover, one can write down such rational solutions (if they exist). For example, let E_i be the elliptic curve given by an equation

$$y_i^2 = x_i^3 - x_i \tag{E$_i$}$$

in $\mathbb{P}^2$, with $i = 1, 2$; let X be a product of E_1 and E_2, and let q be a quadratic form defined by a matrix

$$\begin{pmatrix} 1 & & \\ & -x_1 & \\ & & -x_2 \end{pmatrix}$$

over the function field K of X. Then, by virtue of our results, it can be shown that there exists a K-rational solution of q, and one can explicitly write it down (the explicit form of the solution is found in Example 8.6).

Throughout this article, we always use the étale cohomology, and assume that *the base X is a non-singular, quasi-projective variety over a field k, the integer n is positive, prime to the characteristic p of k, and k contains a primitive* nth *root ζ of unity.*

1. Construction of Azumaya Algebras

In this section, we shall give a construction of an Azumaya algebra of rank n^2 over X from a pair of n-torsion line bundles. Strictly speaking, we shall define a map

$$H^1(X, \mu_n) \times H^1(X, \mu_n) \to H^1(X, PGL_n),$$

which, in the special case $n = 2$, has been given by D. Mumford [9, Sect. 3].

Remark 1.1. Taking cohomology of the sequence (KS), one can interpret $H^1(X, \mu_n)$ as the set of isomorphism classes of couples (L, Φ) where L is an n-torsion line bundle over X and Φ is an isomorphism $\mathcal{O}_X \to L^{\otimes n}$. In case X is complete over an algebraically closed field, it follows that

$$H^1(X, \mu_n) \simeq \mathrm{Pic}(X)_n.$$

In case X is a spectrum of a field K, it follows that

$$K^*/K^{*n} \simeq H^1(K, \mu_n)$$
$$H^2(K, \mu_n) \simeq H^2(K, \mathbb{G}_m)_n.$$

Now, let L and M be n-torsion line bundles over X with isomorphisms $\Phi: \mathcal{O}_X \to L^{\otimes n}$ and $\Psi: \mathcal{O}_X \to M^{\otimes n}$. For a pair of such couples (L, Φ) and (M, Ψ), consider a vector bundle over X:

$$A := \bigoplus_{0 \leqslant i, j \leqslant n-1} L^{\otimes i} \otimes M^{\otimes j}.$$

Using Φ and Ψ, one can define the maps

$$L^{\otimes i} \otimes M^{\otimes j} \otimes L^{\otimes k} \otimes M^{\otimes l} \xrightarrow{\zeta^{jk}} L^{\otimes i} \otimes L^{\otimes k} \otimes M^{\otimes j} \otimes M^{\otimes l}$$
$$\longrightarrow L^{\otimes r} \otimes M^{\otimes s},$$

where $i+k \equiv r$, $j+l \equiv s$ modulo n, and $0 \leqslant r$, $s \leqslant n-1$, and ζ is the primitive nth root of unity. Thus, we obtain an $\mathcal{O}_X$-algebra structure on A.

To investigate a local structure of this algebra A, take an affine open neighborhood U of an arbitrary point in X over which

$$L|_U = \mathcal{O}_U \cdot l \simeq \mathcal{O}_U,$$
$$M|_U = \mathcal{O}_U \cdot m \simeq \mathcal{O}_U,$$

where l, m are generators of L, M over U, respectively. Since both $\Phi(1)|_U$ and $l^{\otimes n}$ generate $L^{\otimes n}|_U$, there exists a unit a in $\Gamma(U, \mathcal{O}_U)$ such that

$$a \cdot \Phi(1)|_U = l^{\otimes n}.$$

Similarly, for M, there exists a unit b in $\Gamma(U, \mathcal{O}_U)$ such that

$$b \cdot \Psi(1)|_U = m^{\otimes n}.$$

Then, we see that the restriction $A|_U$ is isomorphic to an $\mathcal{O}_U$-algebra generated by elements l, m with relations

$$l^n = a, \qquad m^n = b, \qquad lm = \zeta ml.$$

Particularly, in case $n = 2$, $A|_U$ is isomorphic to an $\mathcal{O}_U$-algebra generated by l, m with relations

$$l^2 = a, \qquad m^2 = b, \qquad lm = -ml,$$

namely, a *quaternion algebra* over U. Hence, A is an Azumaya algebra of rank n^2 over X, in particular, a quaternion algebra over X when

$n=2$ (see, e.g., [7, IV, (2.1)]). We denote by $A((L, \Phi), (M, \Psi))$ the algebra A obtained from a pair of the couples (L, Φ), (M, Ψ), and by $P((L, \Phi), (M, \Psi))$ the projective space bundle naturally corresponding to A via the functor of left ideals of A which are subbundles of A of rank n (see, e.g., [1, Sect. 4]). Thus, our construction gives the required map.

Remark 1.2. In case X is a spectrum of a field K, by the isomorphism $H^1(K, \mu_n) \to K^*/K^{*n}$ in Remark 1.1, the couples (L, Φ), (M, Ψ) are assigned to the elements a, b above modulo K^{*n}, respectively. So, in this case, the map A above gives the construction of ordinary cyclic algebras over the field K.

Next, we study the case $n=2$ in detail. In this case, we have another method for constructing projective space bundles of rank 2, namely, *projective line bundles*, from a pair of 2-torsion line bundles as follows. For any 2-torsion line bundles L and M with isomorphisms $\Phi: \mathcal{O}_X \to L^{\otimes 2}$ and $\Psi: \mathcal{O}_X \to M^{\otimes 2}$, let A be the quaternion algebra $A((L, \Phi), (M, \Psi))$, let E be a direct summand $\mathcal{O}_X \oplus L \oplus M$ of A, and let q be a quadratic form on E defined by the reduced norm of A. In other words, the quadratic form q on E is this: We have three global sections

$$\begin{aligned} 1/\iota(1) &\in \Gamma(X, \mathcal{O}_X^{\vee \otimes 2}) \subset \Gamma(X, S^2(E^\vee)) \\ 1/\Phi(1) &\in \Gamma(X, L^{\vee \otimes 2}) \subset \Gamma(X, S^2(E^\vee)) \\ 1/\Psi(1) &\in \Gamma(X, M^{\vee \otimes 2}) \subset \Gamma(X, S^2(E^\vee)), \end{aligned}$$

where ι is a natural isomorphism $\mathcal{O}_X \to \mathcal{O}_X^{\otimes 2}$. Put

$$q := 1/\iota(1) - 1/\Phi(1) - 1/\Psi(1).$$

Then, we obtain a divisor C of $\mathbb{P}(E^\vee)$ defined by the quadratic form q, which is a conic bundle over X.

Now, we locally investigate this bundle C. With the same notations as above, we have an isomorphism

$$E^\vee|_U = \mathcal{O}_U \cdot 1/1 \oplus \mathcal{O}_U \cdot 1/l \oplus \mathcal{O}_U \cdot 1/m \simeq \mathcal{O}_U \oplus \mathcal{O}_U \oplus \mathcal{O}_U,$$

and an expression

$$q|_U = 1/\iota(1) - 1/\Phi(1) - 1/\Psi(1)|_U \simeq \begin{pmatrix} 1 & & \\ & -a & \\ & & -b \end{pmatrix}$$

via the isomorphism $E^\vee|_U \simeq \mathcal{O}_U \oplus \mathcal{O}_U \oplus \mathcal{O}_U$ above, which is nothing but the restriction to $E|_U$ of the reduced norm of A. Hence, the conic bundle C

over X has no singular fibres. Using an étale cover of X associated to a 2-torsion line bundle, for example, L, we see that C is locally trivial over X for the étale topology, namely, a projective line bundle over X.

Thus, we obtain a projective line bundle C from a pair of the couples (L, Φ) and (M, Ψ), which is denoted by $C((L, \Phi), (M, \Psi))$. The vector bundle E used above and the quadratic form q on E defining C are denoted by $E((L, \Phi), (M, \Psi))$ and $q((L, \Phi), (M, \Psi))$, respectively.

By definition, $C((L, \Phi), (M, \Psi))$ is the projective line bundle naturally corresponding to the quaternion algebra $A((L, \Phi), (M, \Psi))$ (see, e.g., [1, Sect. 4] or [13, XIV, Sect. 2, Remark 3, p. 207]), so the bundles $C((L, \Phi), (M, \Psi))$ and $P((L, \Phi), (M, \Psi))$ are isomorphic over X, and we see that our projective space bundles are explicitly given in terms of conic bundles.

Therefore, we get the diagram (FD) in the introduction, which is called the *fundamental diagram* for X.

DEFINITION 1.3. For any elements a and b of $H^1(X, \mu_n)$, the value $d_n(A(a, b)) = d_n(P(a, b))$ is called the *Hilbert symbol* of a and b over X, and denoted by $\{a, b\}_n$.

Remark 1.4. Our Hilbert symbol over X coincides with the classical one when X is a spectrum of a field (see Remark 1.2).

The next proposition follows directly from the exactness of the lower sequence in (FD).

PROPOSITION 1.5. *For any elements a, b of $H^1(X, \mu_n)$, the following conditions are equivalent*:

(1) $P(a, b) \simeq \mathbb{P}(V^\vee)$, *or equivalently*, $A(a, b) \simeq \mathcal{E}nd(V)$ *for some vector bundle V over X*;

(2) $\{a, b\}_n = 0$.

Under the equivalent conditions above, we say that $P(a, b)$, or $A(a, b)$ *comes from* the vector bundle V.

2. COMMUTATIVITY OF THE FUNDAMENTAL DIAGRAM

In this section, we shall verify the commutativity of the fundamental diagram (FD), and give a definition of composition of vector bundles.

LEMMA 2.1. *If X is a spectrum of a field K, then the fundamental diagram* (FD) *for X is commutative.*

Proof. See Remark 1.4 and e.g., [13, XIV, Sect. 2, Proposition 5].

LEMMA 2.2. *Let K be the function field of a general X. Then, the natural map*

$$H^2(X, \mathbb{G}_m) \to H^2(K, \mathbb{G}_m)$$

is injective.

Proof. See, e.g., [7, III, (2.22)].

PROPOSITION 2.3. *Let x and x' be elements of $H^1(X, PGL_n)$, represented by projective space bundles P and P', respectively, or Azumaya algebras A and A', respectively. Then, the following conditions are equivalent:*

(1) *P and P' are birational over X, written $P \sim P'$, or equivalently, A and A' are isomorphic at the generic point of X, written $A \sim A'$;*

(2) $d_n(x) = d_n(x')$.

Proof. This follows from Lemma 2.2 and the fact that the map d_n in (FD) is injective at the generic point of X (see, e.g., [13, X, Sect. 1, Proposition 3]).

COROLLARY 2.4. *For any elements a, a', b and b' of $H^1(X, \mu_n)$, the following conditions are equivalent:*

(1) *$P(a, b) \sim P(a', b')$, or equivalently, $A(a, b) \sim A(a', b')$;*

(2) $\{a, b\}_n = \{a', b'\}_n$.

Now we have

PROPOSITION 2.5. *The fundamental diagram* (FD) *for a general base X is commutative.*

Proof. By the injectivity, Lemma 2.2, we see that the commutativity of (FD) for the generic point Spec K of X implies the commutativity of (FD) for the whole space X. Thus, we have only to show the statement for Spec K, which follows directly from Lemma 2.1.

From Proposition 2.5 above, we get the following corollaries, which will be used in the discussion below.

COROLLARY 2.6. *For any elements a, b and c of $H^1(X, \mu_n)$, we have*

(a) $\{a \otimes b, c\}_n = \{a, c\}_n + \{b, c\}_n$;

(b) $\{a, b \otimes c\}_n = \{a, b\}_n + \{a, c\}_n$;

(c) $\{a, b\}_n + \{b, a\}_n = 0$.

Proof. The required results follow directly from the fact that the cup-product $\cup$ is bilinear and alternating.

COROLLARY 2.7. *For any elements a and b of $H^1(X, \mu_n)$, the following conditions are equivalent:*

(1) *The projective space bundle $P(a, b)$, or equivalently, the Azumaya algebra $A(a, b)$ comes from some vector bundle over X.*

(2) *The cup-product $a \cup b$ in $H^2(X, \mu_n)$ is equal to $c_X(Z)$ for some line bundle Z over X.*

(3) $\{a, b\}_n = 0$.

Proof. Combine Propositions 1.5 and 2.5.

Under the equivalent conditions above, we say that the cup-product $a \cup b$ *comes from* the line bundle Z over X.

Now, we give this

DEFINITION 2.8. For elements a, b, and c of $H^1(X, \mu_n)$, we define the *composition of the pairs* (a, c) and (b, c) to be the pair $(a \otimes b, c)$ in $H^1(X, \mu_n) \times H^1(X, \mu_n)$. Moreover, we define the *composition of the projective space bundles* $P(a, c)$ and $P(b, c)$ to be the bundle $P(a \otimes b, c)$. Furthermore, if $P(a, c)$ and $P(b, c)$ come from vector bundles V_a and V_b, respectively, then we define the *composition of the vector bundles* V_a and V_b to be a vector bundle V_{ab} modulo $\mathrm{Pic}(X)$ such that $P(a \otimes b, c)$ comes from V_{ab}. By virtue of Corollaries 2.6 and 2.7, the existence of the composition V_{ab} is guaranteed. But, one should note that, for isomorphism classes of projective space bundles, or vector bundles, the composition of them are *not* well defined since it depends on the choice of the pairs (a, c) and (b, c). So, we shall specify the pairs of the elements of $H^1(X, \mu_n)$ whenever we use this terminology. Similarly, we define the *composition of the pairs* (a, b) and (a, c) to be the pair $(a, b \otimes c)$ in $H^1(X, \mu_n) \times H^1(X, \mu_n)$, and so on.

Finally, we give a sufficient condition for $\mathrm{Br}(X) = \mathrm{Br}'(X)$ (see [7, IV, (2.9)]), where $\mathrm{Br}'(X)$ is the cohomological Brauer group $H^2(X, \mathbb{G}_m)_{\mathrm{tor}}$ of X.

COROLLARY 2.9. *If the map*

$$H^1(X, \mu_n) \otimes H^1(X, \mu_n) \to H^2(X, \mu_n)$$

defined by the cup-product $\cup$ is surjective, then the set

$$\{\{a, b\}_n \mid a, b \in H^1(X, \mu_n)\}$$

generates the n-torsion part $\mathrm{Br}'(X)_n$. *In particular, we have*

$$\mathrm{Br}(X)_n = \mathrm{Br}'(X)_n.$$

Proof. This follows from Proposition 2.5 and the fact that the image of $H^2(X, \mu_n)$ in $H^2(X, \mathbb{G}_m)$ is equal to $\mathrm{Br}'(X)_n$.

3. Rational Sections and Vector Bundles

In this section, we shall study the relationship between rational sections of a conic bundle C and vector bundles V such that C comes from V.

Lemma 3.1. *For a projective space bundle P over X, the following conditions are equivalent:*

(1) *P comes from a vector bundle over X;*

(2) *P has a rational section over X.*

Proof. See, e.g., [7, III, Exercise 4.24], [12, Proposition 18], or [13, X, Sect. 6, Exercise 1].

Lemma 3.2. *Let P be a projective line bundle over X with projection π, and let ω_π be a relative canonical bundle of π. Then, we have*

(a) *P is isomorphic to a quadratic divisor C of $\mathbb{P}(E^\vee)$ for some vector bundle E of rank 3 over X.*

(b) *Any such E as above is isomorphic to the vector bundle $(\pi_*(\omega_\pi^\vee))^\vee$ modulo $\mathrm{Pic}(X)$, in particular, uniquely determined by P up to tensoring line bundles over X.*

Proof. (a) Put

$$E := (\pi_*(\omega_\pi^\vee))^\vee.$$

Then, according to Grauert's theorem, the direct image $\pi_*(\omega_\pi^\vee)$ is a vector bundle of rank 3, so is its dual E, and we have a natural surjection $\pi^*E^\vee \to \omega_\pi^\vee$ over P. From this surjection, we get a morphism $P \to \mathbb{P}(E^\vee)$. Moreover, we see that this morphism gives an isomorphism from P onto its image C in $\mathbb{P}(E^\vee)$ over X, and C is a quadratic divisor of $\mathbb{P}(E^\vee)$.

(b) For any such vector bundle E as in the statement (a), according to the adjunction formula, we have

$$\pi_*(\mathcal{O}_{\mathbb{P}(E^\vee)}(1) \otimes \mathcal{O}_C) \equiv \pi_*(\omega_\pi^\vee) \qquad \text{modulo } \mathrm{Pic}(X).$$

On the other hand, it is easy to show that

$$E^\vee = \pi_* \mathcal{O}_{\mathbb{P}(E^\vee)}(1) \simeq \pi_*(\mathcal{O}_{\mathbb{P}(E^\vee)}(1) \otimes \mathcal{O}_C),$$

where the projection of $\mathbb{P}(E^\vee)$ over X is also denoted by π. Combining these facts, we get the required result.

PROPOSITION 3.3. *Let K be the function field of X, let C be a projective line bundle over X, and let q be a quadratic form over X which defines the conic bundle C as in Lemma* 3.2. *Then, the following conditions are equivalent*:

(1) *C comes from a vector bundle over X*;

(2) *C has a rational section over X*;

(3) *q has a K-rational solution at the generic point* Spec *K of X.*

Proof. The equivalence follows from Lemma 3.1, where one should note that a closure of a K-rational point of the generic fibre of C over X gives a rational section of C over X.

Now, we prove

THEOREM 3.4. *Let C be a projective line bundle over X, and let E be a vector bundle of rank 3 over X such that C is isomorphic to a quadratic divisor of $\mathbb{P}(E^\vee)$ as in Lemma* 3.2. *Assume that C has a rational section over X, and identify C with the divisor of $\mathbb{P}(E^\vee)$ above. Then*:

(a) *For a rational section of C over X, there exist a unique line bundle S over X and a unique homomorphism $s: S \to E$ satisfying the following conditions*:

(1) *s is injective as a homomorphism of sheaves over X.*

(2) *The zero locus $(s)_0$ of s as a homomorphism of vector bundles has codimension at least 2 in X.*

(3) *The cokernel of s, denoted by V_0, is a torsion-free sheaf of rank 2 over X.*

(4) *The rational map $\mathbb{P}(S^\vee) \to \mathbb{P}(E^\vee)$ defined by s gives a section of C via an isomorphism $X \simeq \mathbb{P}(S^\vee)$, which is defined over the complement $X - (s)_0$ and coincides with the given rational section of C over X.*

Thus, we have an exact sequence over X

$$0 \to S \xrightarrow{s} E \to V_0 \to 0.$$

(b) *Let V be the double dual of V_0. Then, V is a vector bundle of rank 2 over X and the bundle C comes from V.*

Proof. (a) First of all, we note that the conditions (2) and (3) are equivalent since X is locally factorial, and these conditions clearly imply the condition (1). Let U be the maximal open subset of X over which the given rational section can be extended as a morphism, and let i be the inclusion of U into X. We see that there exists a unique rank 1 subbundle

S' of $E|_U$ over U such that the induced map $\mathbb{P}(S'^{\vee}) \to \mathbb{P}(E^{\vee}|_U)$ coincides with the rational section via $U \simeq \mathbb{P}(S'^{\vee})$. Since the complement $X - U$, denoted by Z, has codimension at least 2 in X, there exists a unique line bundle S over X such that $S|_U \simeq S'$. Computing cohomology sheaves with support in Z (see, e.g., [11, II, (1.1.12)]), we see that $S \simeq i_*(S|_U)$ and $E \simeq i_*(E|_U)$, and we get a homomorphism $s: S \to E$ whose restriction $s|_U$ coincides with the inclusion of S' into $E|_U$. Thus, S and s satisfy the required conditions, where one should note that Z coincides with $(s)_0$ because of the maximality of U. Their uniqueness can be easily shown.

(b) In each fibre of $\mathbb{P}(E^{\vee})$ over the open set U, consider the projection from the complement $\mathbb{P}(E^{\vee}) - \mathbb{P}(S^{\vee})$ to $\mathbb{P}(V^{\vee})$ with center $\mathbb{P}(S^{\vee})$. We see that the projection gives an isomorphism between C and $\mathbb{P}(V^{\vee})$ over U. According to Proposition 3.3, there exists a vector bundle W over X such that C comes from W. Thus, from the isomorphism above, we obtain an isomorphism $V|_U \simeq W \otimes L|_U$ for some line bundle L over X. Now, it is easy to show that a dual of a coherent sheaf over X is reflexive; so is the double dual V. Therefore, it follows that

$$V = i_*(V|_U) \simeq i_*(W \otimes L|_U) = W \otimes L$$

(see [loc. cit.]), and C comes from V.

EXAMPLE 3.5. Let L be a 2-torsion line bundle over X with an isomorphism $\Phi: \mathcal{O}_X \to L^{\otimes 2}$. Consider a projective line bundle $P((\mathcal{O}_X, \iota), (L, \Phi))$, denoted by P, where ι is a natural isomorphism $\mathcal{O}_X \to \mathcal{O}_X^{\otimes 2}$. Let E be the vector bundle $E((\mathcal{O}_X, \iota), (L, \Phi))$, and let q be the quadratic form $q((\mathcal{O}_X, \iota), (L, \Phi))$ on E. Then, P is isomorphic to a quadratic divisor C of $\mathbb{P}(E^{\vee})$ defined by q, which is the conic bundle $C((\mathcal{O}_X, \iota), (L, \Phi))$ over X (see Lemma 3.3). Clearly, we have $\{(\mathcal{O}_X, \iota), (L, \Phi)\}_2 = 0$.

According to the local investigation of conic bundles at Section 1, q is represented by a matrix

$$\begin{pmatrix} 1 & & \\ & -1 & \\ & & -f \end{pmatrix} \quad \text{with} \quad f \in K^*$$

at the generic point Spec K of X, which has a K-rational solution $(1 : 1 : 0)$. Now, we define a homomorphism s from a line bundle $\mathcal{O}_X$ to E by $s(\alpha) := (\alpha : \alpha : 0)$. Then, we see that the rational map $\mathbb{P}(\mathcal{O}_X^{\vee}) \to \mathbb{P}(E^{\vee})$ defined by s gives a global section of C over X via $X \simeq \mathbb{P}(\mathcal{O}_X^{\vee})$, and the cokernel V_0 of s is isomorphic to a direct sum $\mathcal{O}_X \oplus L$. In this case, $(s)_0$ is empty, and $V = V_0 = \mathcal{O}_X \oplus L$ with the same notations as above. Thus, according to Theorem 3.4(b), the projective line bundle P comes from the vector bundle $\mathcal{O}_X \oplus L$.

Similarly, we see that both $P((L, \Phi), (\mathcal{O}_X, \imath))$ and $P((L, \Phi), (L, \Phi))$ also come from $\mathcal{O}_X \oplus L$.

PROPOSITION 3.6. *Consider the case $n = 2$. Let L and L' be 2-torsion line bundles over X with isomorphisms $\Phi : \mathcal{O}_X \to L^{\otimes 2}$ and $\Phi' : \mathcal{O}_X \to L'^{\otimes 2}$, and let $\imath$ be a natural isomorphism $\mathcal{O}_X \to \mathcal{O}_X^{\otimes 2}$. Then, the vector bundle $\mathcal{O}_X \oplus L \otimes L'$ is the composition of the vector bundles $\mathcal{O}_X \oplus L$ and $\mathcal{O}_X \oplus L'$ defined by the pairs $((\mathcal{O}_X, \imath), (L, \Phi))$ and $((\mathcal{O}_X, \imath), (L', \Phi'))$.*

Proof. See Definition 2.8 and Example 3.5.

4. COMPOSITION OF VECTOR BUNDLES

We first study geometric meaning of the composition of our projective line bundles (see Definition 2.8). In this section, we always consider the case $n = 2$.

PROPOSITION 4.1. *Let L, L' and M be 2-torsion line bundles over X with isomorphisms $\Phi : \mathcal{O}_X \to L^{\otimes 2}$, $\Phi' : \mathcal{O}_X \to L'^{\otimes 2}$ and $\Psi : \mathcal{O}_X \to M^{\otimes 2}$. Let C and C' be the projective line bundles $C((L, \Phi), (M, \Psi))$ and $C((L', \Phi'), (M, \Psi))$, respectively, let C'' be the composition of C and C', and let E, E', and E'' be the vector bundles $E((L, \Phi), (M, \Psi))$, $E((L', \Phi'), (M, \Psi))$, and $E((L'', \Phi''), (M, \Psi))$, respectively, where we put $(L'', \Phi'') := (L, \Phi) \otimes (L', \Phi')$ in $H^1(X, \mu_2)$. Let $(X : Y : Z)$, $(X' : Y' : Z')$, and $(X'' : Y'' : Z'')$ be the global coordinates of $E = \mathcal{O}_X \oplus L \oplus M$, $E' = \mathcal{O}_X \oplus L' \oplus M$, and $E'' = \mathcal{O}_X \oplus L'' \oplus M$ over X, respectively, and let*

$$\varphi : \mathbb{P}(E^\vee) \times_X \mathbb{P}(E'^\vee) \to \mathbb{P}(E''^\vee)$$

be a rational map defined by $\varphi((X : Y : Z) \times (X' : Y' : Z')) = (X'' : Y'' : Z'')$ with

$$X'' := X \otimes X' + \Psi^{-1} \circ Z \otimes Z'$$

$$Y'' := Y \otimes Y'$$

$$Z'' := X \otimes Z' + Z \otimes X'.$$

Then, we have

(a) *The image of the restriction $\varphi|_{C \times_X C'}$ is dense in C''.*

(b) *The base locus of $\varphi|_{C \times_X C'}$ is contained in a fibre product $H \times_X H'$,* *where H, H' are tautological divisors of $\mathbb{P}(E^\vee)$, $\mathbb{P}(E'^\vee)$ defined by natural inclusions*

$$\mathcal{O}_X \oplus M \to E, \qquad \mathcal{O}_X \oplus M \to E',$$

respectively.

Proof. To prove the statements, we have only to consider the problem at each fibres over X. So, we may assume that X is a spectrum of a field. Then, the required results follow from a direct computation.

Using Proposition 4.1, one can define *the composition of the maps* s in Theorem 3.4(a) in the obvious way, by which we shall define the composition of rational sections of our projective line bundles.

THEOREM 4.2. *With the same notations as above, assume that the bundles C and C' have rational sections over X and the element (M, Ψ) in $H^1(X, \mu_2)$ is not zero (see Example* 3.5). *Let $s = (X : Y : Z)$, $s' = (X' : Y' : Z')$ be the maps $S \to E$, $S' \to E'$ corresponding to the rational sections as in Theorem* 3.4(a), *respectively, let s'' be the composition of s and s', and let V, V' and V'' be the double dual of the cokernels of s, s', and s'', respectively. Then*

(a) *We have*

$$(s'')_0 \subset (s)_0 \cup (s')_0 \cup ((Y)_0 \cap (Y')_0),$$

in particular, $(s'')_0$ is a proper subset of X, and s'' defines a rational map $\mathbb{P}(S^\vee \otimes S'^\vee) \to \mathbb{P}(E''^\vee)$, which gives a rational section of C'' over X via an isomorphism $X \simeq \mathbb{P}(S^\vee \otimes S'^\vee)$.

(b) *If $(s'')_0$ has codimension at least* 2 *in X, then V'' is a vector bundle of rank* 2 *over X, and C'' comes from V''. In other words, the vector bundle V'' is the composition of V and V' defined by the pairs $((L, \Phi), (M, \Psi))$ and $((L', \Phi'), (M, \Psi))$.*

Proof. (a) The assertion follows from the definition of s'' and Proposition 4.1, where we note that neither Y nor Y' is identically zero since (M, Ψ) is not zero.

(b) The required result follows directly from Theorem 3.4.

DEFINITION 4.3. By virtue of Theorem 4.2(a) above, from rational sections of the bundles C and C' over X, we obtain a rational section of C'' over X, which is called the *composition of the rational sections* of C and C' over X (defined by the pairs $((L, \Phi), (M, \Psi))$ and $((L', \Phi'), (M, \Psi))$).

Therefore, using rational sections, we can construct the composition of vector bundles under the conditions above, which will play a key role in Section 7.

Remark 4.4. In the same situation as in Theorem 4.2, consider the spacial case (L, Φ) is equal to either $(\mathcal{O}_X, \iota)$ or (M, Ψ). Then, the quaternion algebras $A((L', \Phi'), (M, \Psi))$ and $A((L'', \Phi''), (M, \Psi))$ are isomorphic over X; so are the bundles C' and C''. In particular, if C' comes from a vector bundle V' over X, then so does C''. Thus, in this case, V' is the composition of $\mathcal{O}_X \oplus L$ and V' defined by the pairs $((L, \Phi), (M, \Psi))$ and $((L', \Phi'), (M, \Psi))$.

5. Cycle Map on a Product of Two Elliptic Curves

In this section, we investigate the cycle map c_X from $\operatorname{Pic}(X)$ to $H^2(X, \mu_n)$ when the base X is a product of two elliptic curves defined over an algebraically closed field. From now on, we assume that *the ground field k is algebraically closed.* For any elliptic curve E, we always fix the unity of group structure of E.

First, we study the Picard group of a product of two elliptic curves.

Lemma 5.1. *Let E_1 and E_2 be elliptic curves, and let R be the group* $\operatorname{Hom}(E_1, \hat{E}_2) = \operatorname{Hom}(E_2, \hat{E}_1)$ *of correspondences between E_1 and E_2. Then, we have an exact sequence*

$$0 \longrightarrow \operatorname{Pic}(E_1) \oplus \operatorname{Pic}(E_2) \xrightarrow{\alpha} \operatorname{Pic}(E_1 \times E_2) \xrightarrow{\beta} R \longrightarrow 0.$$

Proof. First, for a line bundle L_i over E_i, $i = 1, 2$, we define

$$\alpha(L_1 \oplus L_2) := p_1^* L_1 \otimes p_2^* L_2,$$

where p_i is the ith projection of $E_1 \times E_2$. Next, for a line bundle L over $E_1 \times E_2$, we have a line bundle L_1 over E_1 defined by

$$L_1 := L|_{E_1 \times \{e_2\}},$$

where e_2 is the unity of the group E_2. The line bundle $L \otimes p_1^* L_1^\vee$ gives a homomorphism φ from E_2 to $\hat{E}_1$ by the universality of the dual variety $\hat{E}_1$. So, we define

$$\beta(L) := \varphi.$$

Thus, we get homomorphisms α and β. Then, one can easily show that the sequence above is exact.

Next, we recall well-known facts about the cohomology ring of an abelian variety with coefficient in $\mathbb{Z}/n\mathbb{Z}$.

LEMMA 5.2. *Let X be an abelian variety of dimension g. Then, we have*

(a) $H^1(X, \mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})^{\oplus 2g}$;

(b) $H^*(X, \mathbb{Z}/n\mathbb{Z}) = \bigwedge H^1(X, \mathbb{Z}/n\mathbb{Z})$.

PROPOSITION 5.3 (compare [3; 4, Theorem 1; 6, pp. 235–236, A2]). *Let X be an abelian variety. Then, the n-torsion part $\mathrm{Br}(X)_n$ is generated by the set of Hilbert symbols $\{L, M\}_n$ with n-torsion line bundles L and M over X, and we have an exact sequence*

$$0 \longrightarrow \mathrm{Pic}(X)/n\,\mathrm{Pic}(X) \xrightarrow{c_X} H^2(X, \mu_n) \longrightarrow \mathrm{Br}(X)_n \longrightarrow 0$$

Proof. Combine Corollary 2.9 and Lemma 5.2. The exact sequence is induced from the Kummer sequence (KS).

COROLLARY 5.4. *Let X be an abelian variety of dimension g, and let ρ be the rank of the Néron-Severi group of X. Then, the n-torsion part $\mathrm{Br}(X)_n$ is a free $\mathbb{Z}/n\mathbb{Z}$-module of rank $\binom{2g}{2} - \rho$.*

Proof. Combine Lemma 5.2 and Proposition 5.3.

LEMMA 5.5. *Let X be a product of elliptic curves E_1 and E_2. Then, we have a commutative diagram*

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Pic}(E_1) \oplus \mathrm{Pic}(E_2) \arrow[r, "c_{E_1} \oplus c_{E_2}"] \arrow[d] & H^2(E_1, \mu_n) \oplus H^2(E_2, \mu_n) \arrow[d] \\
\mathrm{Pic}(X) \arrow[r, "c_X"] \arrow[d] & H^2(X, \mu_n) \arrow[d] \\
R \arrow[r, "\gamma"] \arrow[d] & H^1(E_1, \mu_n) \otimes H^1(E_2, \mu_n) \arrow[d] \\
0 & 0
\end{tikzcd}$$

with exact rows. Moreover, the top horizontal map is surjective.

Proof. The right vertical sequence is induced from Künneth formula (see, e.g., [7, VI, (8.13)]), so this sequence is exact. Since the top and middle horizontal maps are defined by the cycle maps on E_1, E_2, and X, the upper square is commutative.

The exactness of the left vertical sequence follows from Lemma 5.1. Thus, c_X induces the bottom map γ, and we get the exact commutative diagram above. The later assertion follows from Tsen's theorem (see, e.g., [7, III, (2.22)]).

PROPOSITION 5.6. *Let X be a product of elliptic curves E_1 and E_2. Then, we have an exact sequence*

$$0 \longrightarrow R/nR \xrightarrow{\gamma} H^1(E_1, \mu_n) \otimes H^1(E_2, \mu_n) \longrightarrow \mathrm{Br}(X)_n \longrightarrow 0,$$

where R is the group of correspondences between E_1 and E_2, and γ is the map defined by the cycle map c_X as in Lemma 5.5.

Proof. Combine Proposition 5.3 and Lemma 5.5.

COROLLARY 5.7. *Let X be a product of elliptic curves E_1 and E_2. Then, we have*

$$\operatorname{rank} \mathrm{Br}(X)_n = \begin{cases} 4 & \text{if } E_1, E_2 \text{ are not isogenous} \\ 3 & \qquad\quad \text{isogenous, not of } CM\text{-type} \\ 2 & \qquad\qquad\quad \text{of } CM\text{-type, not supersingular} \\ 0 & \qquad\qquad\qquad\quad \text{supersingular.} \end{cases}$$

Proof. Combine Lemma 5.2 and Proposition 5.6, and use the well-known fact about the ring of endomorphism of an elliptic curve (see, e.g., [5, IV, (4.19); 10, IV, Sect. 22, Second example]).

Now, looking at the meaning of the map γ defined as above, we find that γ is composed of

$$\begin{aligned} R = \mathrm{Hom}(E_2, \hat{E}_1) &\to \mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}(H^1(\hat{E}_1, \mu_n), H^1(E_2, \mu_n)) \\ &\to \mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}(H^1(\hat{E}_1, \mu_n), \mu_n) \otimes_{\mathbb{Z}/n\mathbb{Z}} H^1(E_2, \mu_n) \\ &\to H^1(E_1, \mu_n) \otimes_{\mathbb{Z}/n\mathbb{Z}} H^1(E_2, \mu_n), \end{aligned}$$

where one should note that, by the e_n-pairing over E_1 (see, e.g., [7, V, (2.4)(f)]), there is a canonical isomorphism of $\mathbb{Z}/n\mathbb{Z}$-modules

$$\mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}(H^1(\hat{E}_1, \mu_n), \mu_n) \simeq H^1(E_1, \mu_n).$$

PROPOSITION 5.8. *Let X be a product of elliptic curves E_1 and E_2 with* ith *projection p_i, let L and M be n-torsion line bundles over X, written*

$$L = p_1^* L_1 \otimes p_2^* L_2, \qquad M = p_1^* M_1 \otimes p_2^* M_2,$$

with n-torsion line bundles L_i, M_i *over* E_i, $i = 1, 2$, *and let* γ *be the map defined by the cycle map* c_X *as in Lemma* 5.5. *Then*:

(a) *We have*

$$L \cup M = (L_1 \cup M_1) \oplus (L_2 \cup M_2) \oplus (L_1 \otimes M_2 - M_1 \otimes L_2)$$

via the decomposition

$$H^2(X, \mu_n) = H^2(E_1, \mu_n) \oplus H^2(E_2, \mu_n) \oplus H^1(E_1, \mu_n) \otimes H^1(E_2, \mu_n).$$

(b) *Assume that* L_1, M_1 *are a basis for* $H^1(\hat{E}_1, \mu_n)$, *and let* P_1, Q_1 *be the points of* E_1 *corresponding to* L_1, M_1, *respectively. For a homomorphism* $\varphi\colon E_2 \to \hat{E}_1$ *such that*

$$\hat{\varphi}(P_1) = L_2, \qquad \hat{\varphi}(Q_1) = M_2,$$

we have

$$\gamma(\varphi) = L_1 \otimes M_2 - M_1 \otimes L_2$$

in $H^1(E_1, \mu_n) \otimes H^1(E_2, \mu_n)$.

Proof. (a) This is obvious.

(b) From the meaning of the map γ, one can easily compute the value $\gamma(\varphi)$ in $H^1(E_1, \mu_n) \otimes H^1(E_2, \mu_n)$.

Remark 5.9. Using Proposition 5.6 and this Proposition 5.8, one can easily compute the relations on the set of generators of the group $\mathrm{Br}(X)_n$ of a product X of two elliptic curves (see Example 8.4).

Finally, we have

THEOREM 5.10. *With the same notations as above, the following conditions are equivalent*:

(1) *The projective space bundle* $P(L, M)$, *or equivalently, the Azumaya algebra* $A(L, M)$ *comes from some vector bundle over* X.

(2) *The elements* $L_1 \otimes M_2 - M_1 \otimes L_2$ *in* $H^1(E_1, \mu_n) \otimes H^1(E_2, \mu_n)$ *is equal to* $\gamma(\varphi)$ *for some correspondence* φ *between* E_1 *and* E_2.

(3) $\{L, M\}_n = 0$.

Proof. Combine Corollary 2.7 and Propositions 5.6 and 5.8(a).

Under the equivalent conditions above, we say that the element $L_1 \otimes M_2 - M_1 \otimes L_2$ *comes from* the correspondence φ between E_1 and E_2. In Section 7, we shall explain the relation between the correspondences and the vector bundles above.

As an application of Theorem 5.10, we obtain an elementary, concrete example of projective space bundles which do not come from any vector bundles (see also Example 8.4). Such an example in the case over a complex number field $\mathbb{C}$ has been given by J.-P. Serre [12, 6.4].

EXAMPLE 5.11. With the same notations as above, assume that E_1 and E_2 are not isogenous and $L \cup M$ is not zero. Then, it follows from Theorem 5.10 that the projective space bundle $P(L, M)$ does not come from any vector bundle. Note that, in case $n = 2$, $P(L, M)$ is explicitly given in terms of a conic bundle.

6. SOME PROPERTIES

In this section, we shall study some properties of our projective space bundles over an abelian variety, which will be used in our discussion below. First, we have

PROPOSITION 6.1. *Let X be an abelian variety, and let L and M be n-torsion line bundles over X. Then, the projective space bundle $P(L, M)$ is homogeneous. In particular, if $P(L, M)$ comes from a vector bundle V over X, then V is homogeneous up to tensoring line bundles over X.*

Proof. Since the Azumaya algebra $A(L, M)$ is homogeneous (with its algebra structure), so is $P(L, M)$.

A vector bundle V over an abelian variety is called *semi-homogeneous* if V is homogeneous up to tensoring line bundles as in Proposition 6.1 (see [8, (5.2)]).

Next, we discuss simple projective space bundles which come from vector bundles, where a projective space bundle P over X is called *simple* if the group scheme of automorphisms of P over X is discrete (see [14, (1.2)]).

LEMMA 6.2. *Let X be complete, let V be a vector bundle over X, and let P be a projective space bundle over X which comes from V. Then, the following conditions are equivalent*:

(1) *P is simple*;

(2) $\Gamma(X, \mathcal{E}nd(V)) \simeq k$.

Proof. See [14, (1.5)].

Under the equivalent conditions above, the vector bundle V is called *simple* (see also [11, I, (4.1.1)]).

LEMMA 6.3. *Let M be a finitely generated free $\mathbb{Z}/n\mathbb{Z}$-module, and let a and b be elements of M. Then, the following conditions are equivalent:*

(1) *If $ia + jb = 0$ in M with integers i and j, then $i \equiv j \equiv 0$ modulo n.*

(2) *The exterior product $a \wedge b$ has order n* in $\bigwedge^2 M$.

Proof. One can easily prove this result.

PROPOSITION 6.4. *Let X be an abelian variety, and let L and M be n-torsion line bundles over X. Then, the following conditions are equivalent:*

(1) $\Gamma(X, A(L, M)) \simeq k$.

(2) *The cup-product $L \cup M$ has order n in $H^2(X, \mu_n)$.*

Proof. Combine Lemmas 5.2 and 6.3.

Now, for our projective space bundle over an abelian variety which comes from a vector bundle, we give its criterion to be simple, in terms of the cup-product of the pair of torsion line bundles.

COROLLARY 6.5. *Let X be an abelian variety, and let L and M be n-torsion line bundles over X. Assume that the projective space bundle $P(L, M)$ comes from a vector bundle V over X. Then, the following conditions are equivalent:*

(1) *$P(L, M)$, or equivalently, V is simple.*

(2) *The cup-product $L \cup M$ has order n in $H^2(X, \mu_n)$.*

Proof. Combine Lemma 6.2 and Proposition 6.4, where one should note that

$$\mathscr{E}nd(V) \simeq A(L, M).$$

Finally, we have

PROPOSITION 6.6. *Let X be an abelian variety of dimension g, and let L and M be n-torsion line bundles over X. For an integer d, the following conditions are equivalent:*

(1) *The projective space bundle $P(L, M)$ is a pull-back from an abelian variety of dimension d.*

(2) *Both L and M are pull-backs from an abelian variety of dimension d.*

Proof. The condition (2) clearly implies the condition (1). We have only to show the converse. Assume that there exist an abelian variety Y of dimension d, a homomorphism $\Psi: X \to Y$, and an Azumaya algebra A over

Y such that $A(L, M)$ is isomorphic to the pull-back Ψ^*A from Y. Then, to deduce the condition (2) for the line bundles L and M, we may assume that $1 \leqslant d \leqslant g$, and Ψ is surjective. Let K be the kernel of Ψ, and let K_0 be the connected component of K containing the unity of the group X. It follows that $A(L, M)|_K$ is a trivial vector bundle; so is $A(L, M)|_{K_0}$. Therefore, by the definition of $A(L, M)$, we see that both $L|_{K_0}$ and $M|_{K_0}$ are also trivial. According to Grauert's theorem, this means that both L and M are pull-backs of some line bundles over the quotient X/K_0. Since the quotient K/K_0 is reduced and finite, the group scheme X/K_0 is an abelian variety of dimension d.

7. Vector Bundles over a Product of Two Elliptic Curves

This section contains the main result of this article. From now on, we always consider the case $n = 2$. So, the characteristic p is not 2.

We start with this

Example 7.1. Let X be an elliptic curve, and let P_∞ be the point of X corresponding to the unity of the group X. For any 2-torsion line bundles L and M over X such that the cup-product $L \cup M$ is not zero, let P_0 and P_1 be the points of X corresponding to them, respectively, where we note that the conditions $L \cup M \neq 0$ and $P_0 \neq P_1$ are equivalent.

It follows from Tsen's theorem that the Hilbert symbol $\{L, M\}_2$ is zero, In other words, the projective line bundle $C(L, M)$ has a rational section over X, and comes from some vector bundle over X.

We here construct a global section of $C(L, M)$ and construct the vector bundle over X. One may assume that X is given by an equation

$$y^2 = x(x-1)(x-\lambda) \qquad \text{with} \quad \lambda \neq 0, 1$$

in $\mathbb{P}^2$ such that P_∞ is the point at infinity and P_0, P_1 have coordinates $(0, 0)$, $(1, 0)$, respectively. Let P_λ be the point of X with coordinate $(\lambda, 0)$. In terms of the group structure of X, we have that $P_0 + P_1 = P_\lambda$.

Now, for such a pair (L, M), according to the local investigation of conic bundles over X at Section 1, the quadratic form $q(L, M)$ on the vector bundle $E(L, M)$, denoted by E, is represented by a matrix

$$\begin{pmatrix} 1 & & \\ & -x & \\ & & -(x-1) \end{pmatrix}$$

at the generic point Spec K of X. Clearly, it has a K-rational solution $(1:1:i)$, with $i^2=-1$. By the ratio $(1:1:i)$, we embed the line bundle $\mathcal{O}_X(-P_\infty)$ into E as a subbundle: We define a map s from $\mathcal{O}_X(-P_\infty)$ to $E=\mathcal{O}_X \oplus L \oplus M$ by $s(\alpha):=(\alpha:\alpha:i\alpha)$. Then, we find that s gives a global section of $C(L, M)$ over X. According to Theorem 3.4(b), with the same notations as there, $C(L, M)$ comes from the cokernel $V_0=V$ of s, where $(s)_0$ is empty; in other words, s is an injection of vector bundles, so its cokernel V_0 is already locally free. From the exact sequence of vector bundles over X

$$0 \longrightarrow \mathcal{O}_X(-P_\infty) \xrightarrow{s} E \longrightarrow V \longrightarrow 0,$$

we find that the vector bundle V is indecomposable, of rank 2, with the first Chern class P_λ, where one should note that P_λ is the point of X corresponding to the 2-torsion line bundle $L \otimes M$. Now, according to M. F. Atiyah [2, II, Theorem 7], such a vector bundle V is characterized by the properties above. In this article, a vector bundle V over an elliptic curve X is called *of type Atiyah* (*determined by a point* P of X) if V is indecomposable, of rank 2 and degree 1 (whose first Chern class is represented by the point P). Using the characterization above, we see that a vector bundle of type Atiyah is semi-homogeneous (see [2, II, Corollary, p. 434]), which follows also from Proposition 6.1. On the other hand, it is well known that a vector bundle of type Atiyah is simple (see [2, III, Sect. 2, Lemma 22]), which follows also from Corollary 6.5.

Now, we study what vector bundle comes to our quaternion algebra (or, projective line bundle) obtained from a pair of 2-torsion line bundles over a product of two elliptic curves.

Let X be a product of elliptic curves E_1 and E_2. For any 2-torsion line bundles L and M over X, there exist 2-torsion line bundles L_i and M_i over E_i, with $i=1, 2$, such that

$$L=p_1^*L_1+p_2^*L_2, \qquad M=p_1^*M_1+p_2^*M_2,$$

where p_i is the ith projection from X to E_i, and the tensor products of line bundles are written additively.

Now, assume that the Hilbert symbol $\{L, M\}_2$ is zero. It follows from Theorem 5.10 that the element $L_1 \otimes M_2 - M_1 \otimes L_2$ in $H^1(E_1, \mu_2) \otimes H^1(E_2, \mu_2)$ comes from a correspondence φ between E_1 and E_2:

$$\gamma(\varphi)=L_1 \otimes M_2 - M_1 \otimes L_2.$$

Using φ, we shall construct the vector bundle which comes to the quaternion algebra $A(L, M)$ (or, the projective line bundle $C(L, M)$).

In order to do this, we use Examples 3.5 and 7.1 and Theorem 4.2. Essentially, we first find a rational solution of the quadratic form $q(L, M)$ defining $C(L, M)$ over X, secondly we construct the rational section of $C(L, M)$, and thirdly we construct the required vector bundle.

First, we consider the case the cup-product $L \cup M$ is zero. Then, we see that there exists a non-trivial 2-torsion line bundle N over X such that L and M are isomorphic to some multiples of N, so they are isomorphic to either $\mathcal{O}_X$ or N. Therefore, according to Example 3.5, $C(L, M)$ comes from a trivial vector bundle $V = \mathcal{O}_X \oplus \mathcal{O}_X$ if both L and M are trivial, and $C(L, M)$ comes from the vector bundle $V = \mathcal{O}_X \oplus N$ otherwise. Any way, $C(L, M)$ comes from either $\mathcal{O}_X \oplus L$ or $\mathcal{O}_X \oplus M$.

Next, we consider the case $L_1 \cup M_1$ is not zero. Then, identifying E_1 with its dual, using Lemma 5.2 and Proposition 5.8(b), we find that

$$\varphi^* L_1 = L_2, \qquad \varphi^* M_1 = M_2.$$

Let φ_1 be the composition of $1_{E_1} \times \varphi$ with the group law of E_1. We call Φ_1 the *homomorphism from X to E_1 defined by the correspondence* φ between E_1 and E_2. Then, we see that

$$\Phi_1^* C(L_1, M_1) = C(L, M).$$

According to Example 7.1, $C(L_1, M_1)$ comes from a vector bundle V_1 of type Atiyah over E_1 since we have $L_1 \cup M_1 \neq 0$. Therefore, our bundle $C(L, M)$ comes from the pull-back $\Phi_1^* V_1$ of V_1 by the homomorphism Φ_1. In case $L_2 \cup M_2$ is not zero, similarly, $C(L, M)$ comes from a pull-back $\Phi_2^* V_2$ of a vector bundle V_2 of type Atiyah over E_2 by the homomorphism Φ_2 from X to E_2 defined by φ. In each case, according to Example 7.1, the first Chern class of V_i above is represented by the point of E_i corresponding to the 2-torsion line bundle $L_i + M_i$.

Finally, we consider the case both $L_1 \cup M_1$ and $L_2 \cup M_2$ are zero but $L_1 \otimes M_2 - M_1 \otimes L_2$ is not zero. For each index i, the condition $L_i \cup M_i = 0$ implies that there exists a non-trivial 2-torsion line bundle N_i over E_i such that

$$L_i = a_i N_i, \qquad M_i = b_i N_i \qquad \text{with} \quad a_i, b_i \in \mathbb{Z}/2\mathbb{Z}.$$

Thus, we have

$$\gamma(\varphi) = L_1 \otimes M_2 - M_1 \otimes L_1 = (a_1 b_2 - b_1 a_2)\, N_1 \otimes N_2.$$

Taking account of the condition $L_1 \otimes M_2 - M_1 \otimes L_2 \neq 0$, we have

$$a_1 b_2 - b_1 a_2 = 1.$$

Then, we find that the matrix $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ is one of six matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If the matrix is equal to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then we see that $C(L, M)$ is the composition of $C(p_1^*N_1, p_1^*N_1)$ and $C(p_1^*N_1, p_2^*N_2)$. Therefore, according to Remark 4.4, $C(L, M)$ is isomorphic to $C(p_1^*N_1, p_2^*N_2)$ over X, which is the one corresponding to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

In this way, we find that each of the bundles corresponding to the former four matrices is isomorphic to one of the bundles corresponding to the latter two matrices. On the other hand, one can easily show that the bundles corresponding to the latter two matrices are isomorphic over X. Therefore, we conclude that all the bundles $C(L, M)$ in this case are isomorphic to $C(p_1^*N_1, p_2^*N_2)$. For the projective line bundles of this type, we have

PROPOSITION 7.2. *With the same notations as above, let N_i be a non-trivial 2-torsion line bundle over E_i, $i = 1, 2$. Assume that the Hilbert symbol $\{p_1^*N_1, p_2^*N_2\}_2$ is equal to zero. Then, we have*

(a) *The element $N_1 \otimes N_2$ in $H^1(E_1, \mu_2) \otimes H^1(E_2, \mu_2)$ comes from a correspondence φ between E_1 and E_2.*

(b) *For each index i, if N_i' is a 2-torsion line bundle over E_i such that N_i and N_i' are a basis for $H_1(E_i, \mu_2)$, and V_i is a vector bundle of type Atiyah over E_i determined by the point corresponding to a 2-torsion line bundle $N_i + N_i'$, then $C(p_1^*N_1, p_2^*N_2)$ comes from the composition of $\Phi_i^*V_i$ and $p_i^*V_i$, where Φ_i is the homomorphism from X to E_i defined by φ, and the composition above is defined by the pairs*

$$(p_1^*N_1, p_1^*N_1' + p_2^*N_2) \quad \text{and} \quad (p_1^*N_1, p_1^*N_1'),$$
$$(p_1^*N_1 + p_2^*N_2', p_2^*N_2) \quad \text{and} \quad (p_2^*N_2', p_2^*N_2),$$

respectively.

(c) *The composition above is constructed as described in Theorem* 4.2(b).

Proof. (a) See Theorem 5.10.

(b) We prove the statement in case $i = 1$. We see that $C(p_1^*N_1, p_2^*N_2)$ is the composition of $C(p_1^*N_1, p_1^*N_1' + p_2^*N_2)$ and $C(p_1^*N_1, p_1^*N_1')$, and that

$$C(p_1^*N_1, p_1^*N_1') = p_1^*C(N_1, N_1').$$

On the other hand, identifying the elliptic curve E_1 with its dual, using Lemma 5.2 and Proposition 5.8(b), we find that

$$\varphi^* N_1 = \mathcal{O}_{E_2}, \qquad \varphi^* N'_1 = N_2.$$

Therefore, we see that

$$C(p_1^* N_1, p_1^* N'_1 + p_2^* N_2) = \Phi_1^* C(N_1, N'_1).$$

Now, according to Example 7.1, we have an exact sequence of vector bundles over E_1

$$0 \longrightarrow S \xrightarrow{s=(X:Y:Z)} E \longrightarrow V_1 \longrightarrow 0,$$

where E is the vector bundle $E(N_1, N'_1) = \mathcal{O}_{E_1} \oplus N_1 \oplus N'_1$, and $C(N_1, N'_1)$ comes from the vector bundle V_1 of type Atiyah over E_1 determined by the point corresponding to a 2-torsion line bundle $N_1 + N'_1$. Thus, $C(p_1^* N_1, p_2^* N_2)$ comes from the composition of $\Phi_1^* V_1$ and $p_1^* V_1$.

(c) Using Theorem 4.2(a), we shall show that the composition of $\Phi_1^* V_1$ and $p_1^* V_1$ is constructed as described in Theorem 4.2(b). Corresponding to $\Phi_1^* V_1$ and $p_1^* V_1$, we have two exact sequences of vector bundles over X

$$0 \longrightarrow \Phi_1^* S \xrightarrow{\Phi_1^* s} \Phi_1^* E \longrightarrow \Phi_1^* V_1 \longrightarrow 0$$

and

$$0 \longrightarrow p_1^* S \xrightarrow{p_1^* s} p_1^* E \longrightarrow p_1^* V_1 \longrightarrow 0,$$

respectively. For $\Phi_1^* s$ and $p_1^* s$, define a map s'' to be the composition of them (see Section 4), we obtain an exact sequence over X

$$0 \longrightarrow \Phi_1^* S + p_1^* S \xrightarrow{s''} E(p_1^* N_1, p_2^* N_2) \longrightarrow V''_0 \longrightarrow 0,$$

where V''_0 is the cokernel of s''. Now, we shall show the zero locus $(s'')_0$ has codimension at least 2 in X. The zero locus $(\Phi_1^* Y)_0$ consists of a fibre of Φ_1 since $(Y)_0$ is one point of E_1. Similarly, $(p_1^* Y)_0$ consists of a fibre of p_1. Therefore, both $(\Phi_1^* Y)_0$ and $(p_1^* Y)_0$ are isomorphic to E_2, in particular, irreducible, of codimension 1 in X. On the other hand, the zero loci $(\Phi_1^* Y)_0$ and $(p_1^* Y)_0$ do not coincide since the correspondence φ is not zero. Thus, the intersection $(\Phi_1^* Y)_0 \cap (p_1^* Y)_0$ has codimension at least 2 in X. It follows from Theorem 4.2(a) that $(s'')_0$ has codimension at least 2, where one should note that both $(\Phi_1^* s)_0$ and $(p_1^* s)_0$ are empty. So, we can apply Theorem 4.2(b) to our case, and we see that the double dual of V''_0 is a vector bundle over X, which is the composition of $\Phi_1^* V_1$ and $p_1^* V_1$.

Thus, we have proved

THEOREM 7.3. *Let X be a product of elliptic curves E_1 and E_2 with* ith *projection p_i, let L and M be 2-torsion line bundles over X, written*

$$L = p_1^* L_1 + p_2^* L_2, \qquad M = p_1^* M_1 + p_2^* M_2,$$

with n-torsion line bundles L_i, M_i over E_i, $i = 1, 2$, and let P_i be the 2-torsion point of E_i corresponding to a line bundle $L_i + M_i$, $i = 1, 2$. Assume that the Hilbert symbol $\{L, M\}_2$ is equal to zero, and let Φ_i be the homomorphism from X to E_i defined by the correspondence between E_1 and E_2 which comes to the element $L_1 \otimes M_2 - M_1 \otimes L_2$ in $H^1(E_1, \mu_2) \otimes H^1(E_2, \mu_2)$. Then, we have

(a) *In case the cup-product $L \cup M$ is zero, the quaternion algebra $A(L, M)$ comes from either $\mathcal{O}_X \oplus L$ or $\mathcal{O}_X \oplus M$, corresponding to whether L is non-trivial or not, or whether M is trivial or not.*

(b) *In case $L_i \cup M_i$ is not zero for some index i, let V_i be a vector bundle of type Atiyah over E_i determined by P_i. Then, $A(L, M)$ comes from the pull-back $\Phi_i^* V_i$.*

(c) *In case both $L_1 \cup M_1$ and $L_2 \cup M_2$ are zero but $L_1 \otimes M_2 - M_1 \otimes L_2$ is not zero, let V_i' be a vector bundle of type Atiyah over E_i determined by a non-zero 2-torsion point other than P_i. Then, $A(L, M)$ comes from the composition of $\Phi_i^* V_i'$ and $p_i^* V_i'$, which is constructed as in Theorem* 4.2(b). *In this case, $A(L, M)$ is uniquely determined by the value $L_1 \otimes M_2 - M_1 \otimes L_2$.*

COROLLARY 7.4. *With the same notations as above, if the Hilbert symbol $\{L, M\}_2$ is equal to zero, then the quaternion algebra $A(L, M)$ comes from one of the vector bundles of the following three types:*

(1) *a direct sum $\mathcal{O}_X \oplus L$ or $\mathcal{O}_X \oplus M$;*

(2) *a pull-back of a vector bundle of type Atiyah over either E_1 or E_2 by a morphism defined by $L \cup M$, which is semi-homogeneous and simple;*

(3) *a composition of vector bundles of type* (2) *above, which is semi-homogeneous and simple.*

Proof. See Proposition 6.1, Corollary 6.5, and Theorem 7.3.

Remark 7.5. For a vector bundle V of type (3) in Corollary 7.4, we have both examples, such that V is a pull-back of a vector bundle over some elliptic curve, and such that V is not any pull-back of any vector bundle over any elliptic curve (see Example 8.5).

8. Examples

Throughout this section, we consider the case $n=2$, so that the characteristic p is not 2. We shall discuss some examples over a product of two elliptic curves E given by the equation

$$y^2 = x^3 - x$$

in $\mathbb{P}^2$.

First, we fix some notations and state some elementary facts on the elliptic curve E. Let P_∞ be the point of E at infinity. Considering P_∞ as a unity, define a group structure on E. Via an isomorphism from E to its dual defined by P_∞, we sometimes identify them. Let P_{-1}, P_0, and P_1 be the points of E with coordinates $(-1, 0)$, $(0, 0)$, and $(1, 0)$, respectively. Let L and M be the 2-torsion line bundles over E corresponding to P_0 and P_1, respectively, which form a basis for the group $H^1(E, \mu_2)$.

Computing the Hasse invariant, we have

Lemma 8.1. *E is supersingular if and only if $p \equiv 3$ modulo 4.*

Let R be the ring of endomorphisms of E, and let $\imath$ be the endomorphism of E defined by $\imath(x, y) := (-x, iy)$, with $i^2 = -1$. It clearly follows that

$$\begin{aligned} &\imath^2 + 1 = 0, \\ &\hat{\imath}(P_0) = L, \qquad \hat{\imath}(P_1) = L + M. \end{aligned} \tag{1}$$

Moreover, we have

Lemma 8.2. *If E is not supersingular, then R is freely generated by 1 and $\imath$ as a $\mathbb{Z}$-module.*

Proof. See, e.g., [5, IV, (4.19); 10, IV, Sect. 22, Second example].

If, on the contrary, E is supersingular, then R is a maximal order of the quaternion division algebra $R \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}$ (see, e.g., [loc. cit.]). To get a typical example of a funny phenomenon in this case, we assume $p = 3$ (see Lemma 8.1). Then, let η be an endomorphism of E defined by $\eta(x, y) := (x + 1, y)$. We have

$$\begin{aligned} &\eta^2 + \eta + 1 = 0, \\ &\hat{\eta}(P_0) = L + M, \qquad \hat{\eta}(P_1) = L. \end{aligned} \tag{2}$$

Moreover, we find the following relations

$$\begin{aligned} \imath\eta &= \eta^2\imath, & \imath\eta\imath\eta + 1 &= 0, \\ \eta\imath &= \imath\eta^2, & \eta\imath\eta\imath + 1 &= 0. \end{aligned}$$

Remark 8.3. Furthermore, one can easily show that R is freely generated by 1, ι, η, and $\iota\eta$ as a $\mathbb{Z}$-module.

Now, let E_1 and E_2 be two copies of E, and let X be a product of E_1 and E_2 with ith projection p_i.

First, we shall discuss about the 2-torsion part $\mathrm{Br}(X)_2$ of the Brauer group $\mathrm{Br}(X)$.

EXAMPLE 8.4 (for Remark 5.9). According to Corollary 5.7, if E is supersingular, then $\mathrm{Br}(X)_2$ is zero. So, we assume that E is not supersingular. Again according to Corollary 5.7, in this case, $\mathrm{Br}(X)_2$ is a $\mathbb{Z}/2\mathbb{Z}$-module of rank 2.

Here, we shall find a free generator of $\mathrm{Br}(X)_2$ over $\mathbb{Z}/2\mathbb{Z}$. By virtue of Proposition 5.6 and Lemma 8.2, we see that $\mathrm{Br}(X)_2$ is isomorphic to a $\mathbb{Z}/2\mathbb{Z}$-module generated by $L\otimes L$, $L\otimes M$, $M\otimes L$ and $M\otimes M$ with relations $\gamma(1)=\gamma(\iota)=0$. Using Proposition 5.8(b) and the equalities (1), we have

$$\{p_1^*L, p_2^*M\}_2=\{p_1^*M, p_2^*L\}_2 \tag{3}$$

$$\{p_1^*L, p_2^*L\}_2=0. \tag{4}$$

Thus, $\mathrm{Br}(X)_2$ is freely generated by $\{p_1^*L, p_2^*M\}_2=\{p_1^*M, p_2^*L\}_2$ and $\{p_1^*M, p_2^*M\}_2$ over $\mathbb{Z}/2\mathbb{Z}$. According to Corollary 2.4, the equality (3) means that the quaternion algebras $A(p_1^*L, p_1^*M)$ and $A(p_1^*M, p_2^*L)$ are isomorphic at the generic point of X. According to Proposition 1.5, the equality (4) means that $A(p_1^*L, p_2^*L)$ comes from some vector bundle over X. We note that $A(p_1^*L, p_2^*M)$, $A(p_1^*M, p_2^*L)$, and $A(p_1^*M, p_2^*M)$ do not come from any vector bundles over X.

For any characteristic p except 2, by Example 8.4 above, we see that the quaternion algebra $A(p_1^*L, p_2^*L)$ comes from some vector bundle over X.

In the next example, we discuss about this algebra and the vector bundle.

EXAMPLE 8.5 (for Remark 7.5). According to Theorem 7.3, $A(p_1^*L, p_2^*L)$ comes from a vector bundle of type (3) in Corollary 7.4. We shall show that: In case $p=3$, $A(p_1^*L, p_2^*L)$ comes from a pull-back of a vector bundle over an elliptic curve; In case $p=0$, $A(p_1^*L, p_2^*L)$ does not come from any pull-back of any vector bundle over any elliptic curve.

First, assume $p=3$. Let ψ be the endomorphism $\iota+\eta+\iota\eta$ of E, and define a homomorphism $\Psi\colon X\to E$ to be the composition of $\psi\times\iota\psi$ with the group law of E. Using the equalities (1) and (2), we find that

$$\Psi^*A(L+M, M)=A(p_1^*L, p_2^*L).$$

According to Example 7.1, $A(L+M, M)$ comes from a vector bundle V_3

over E which is of type Atiyah determined by the point P_0. Thus, our algebra $A(p_1^*L, p_2^*L)$ comes from the pull-back $\Psi^* V_3$.

Next, assume $p=0$. In order to prove our claim, by Proposition 6.6, we have only to show that both line bundles p_1^*L and p_2^*L are not pull-backs of any line bundles over any elliptic curve. In this case, we may assume that the ground field k is a complex number field $\mathbb{C}$, and our elliptic curve E is given by

$$E=\mathbb{C}^1/\Gamma, \qquad \Gamma=\mathbb{Z}\cdot 1\oplus\mathbb{Z}\cdot i,$$

with $i^2=-1$ (see, e.g., [5, IV, (4.20.1)]). Hence, it follows

$$X=\mathbb{C}^2/\Gamma\times\Gamma.$$

Identifying X with its dual, the line bundles p_1^*L and p_2^*L correspond to the vectors $((1+i)/2, 0)$ and $(0, (1+i)/2)$ of $\mathbb{C}^2$ modulo $\Gamma\times\Gamma$, respectively. Now, assume that both line bundles are pull-backs of some line bundles over an elliptic curve. Then, there should exist a 1-dimensional vector subspace of $\mathbb{C}^2$ which contains both $((1+i)/2, 0)$ and $(0, (1+i)/2)$ modulo $\Gamma\times\Gamma$. Therefore, we have

$$\det\begin{pmatrix} \dfrac{1+i}{2}+a & b \\ c & \dfrac{1+i}{2}+d \end{pmatrix}=0$$

for some elements a, b, c, and d of Γ. It follows that the complex number $(1+i)/2$ is integral over $\Gamma=\mathbb{Z}[i]$. This contradicts the fact that the ring $\mathbb{Z}[i]$ of Gaussian integers is integrally closed in its quotient field $\mathbb{Q}(i)$. Hence, our algebra $A(p_1^*L, p_2^*L)$ does not come from any pull-back of any vector bundle over any elliptic curve.

Finally, we refer to a rational solution of a quadratic form over the function field K of X. Chasing the construction of the vector bundles at Section 7, one can find a K-rational solution of *all* the quadratic form $q(L, M)$ with 2-torsion line bundles L and M over X.

EXAMPLE 8.6. Let q be the quadratic form $q(p_1^*L, p_2^*L)$. According to the local investigation of conic bundles at Section 1, q is represented by a matrix

$$\begin{pmatrix} 1 & & \\ & -x_1 & \\ & & -x_2 \end{pmatrix}$$

at the generic point Spec K of X, where (x_i, y_i) is the affine coordinate of E_i in $\mathbb{P}^2$, with $i = 1, 2$. This quadratic form q defines the conic bundle $C(p_1^*L, p_2^*L)$, which comes from a vector bundle of type (3) in Corollary 7.4. So, chasing the construction of vector bundles of this type, making a calculation, we find a solution $(X : Y : Z)$ of q, where

$$X := \frac{1-i}{2}\left(x_1^2 x_2 + \frac{i}{2} y_1^2\right) + x_1\left(\frac{1+i}{4}(x_1^2+1)x_2 - \frac{i}{2} y_1 y_2\right)$$

$$Y := \frac{1+i}{4} y_1(x_1+i)(x_2-1) - \frac{i}{2} y_2 x_1(x_1-i)$$

$$Z := \frac{1-i}{2}\left(x_1^2 x_2 + \frac{i}{2} y_1^2\right) + \frac{x_1}{x_2}\left(\frac{1+i}{4}(x_1^2+1)x_2 - \frac{i}{2} y_1 y_2\right)$$

with $i^2 = -1$.

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